Max-and-Min Fractional Programming for Communications and Sensing

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***Abstract*—There is considerable interest in the use of fractional programming (FP) for the communication system design because many problems in this area are fractionally structured. Notably, max-FP and min-FP are not interchangeable in general if there are multiple ratios, so the two types of FP are often dealt with separately in the existing literature. As a result, an FP method for maximizing the signal-to-interference-plus-noise ratios (SINRs) typically cannot be used for minimizing the Cramér-Rao bounds (CRBs). In contrast, this work proposes a unified approach that bridges the gap between max-FP and min-FP. Particularly, we examine the theoretical basis of this unified approach from a minorization-maximization (MM) perspective, and in return obtain a matrix extension of this new FP technique. Moreover, this work presents two application cases: (i) joint radar sensing and (ii) multi-cell secure transmission, neither of which can be efficiently addressed by the existing FP tools.**

***Index Terms*—Multi-ratio fractional programming (FP), max-FP, min-FP, secure transmission, Cramér-Rao bound, sensing.**

I. Introduction

In a broad sense, fractional programming (FP) is a class of mathematical optimizations whose main components are ratio terms. The studies in the literature are mostly focused on the *concave-convex sum-of-ratios maximization* problem

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \sum_{n=1}^{N} \frac{A_n(\boldsymbol{x})}{B_n(\boldsymbol{x})}, \tag{1}$$

where each $A_n(\boldsymbol{x})$ is a concave nonnegative function and each $B_n(\boldsymbol{x})$ is a convex positive function over the nonempty convex constraint set $\mathcal{X}$, or its minimization counterpart

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \sum_{n=1}^{N} \frac{A_n(\boldsymbol{x})}{B_n(\boldsymbol{x})} \tag{2}$$

with the convexity and the concavity assumptions exchanged between $A_n(\boldsymbol{x})$ and $B_n(\boldsymbol{x})$, while the rest settings remain the same. Notice that neither (1) nor (2) is a convex problem in general even if the objective function comprises only one ratio.

A classical result is that problem (1) with $N = 1$, i.e., the single-ratio max-FP, can be efficiently solved via the Charnes-Cooper algorithm [1], [2] or Dinkelbach's algorithm [3]. In particular, since minimizing a ratio is equivalent to maximizing its reciprocal, problem (2) with $N = 1$ can be readily recast to (1) by flipping the ratio, and hence the above traditional algorithms work for the single-ratio min-FP as well. However, when there are multiple ratios, it is difficult to rewrite min-FP as max-FP, and vice versa, so these two types of FP are dealt with separately in most previous studies. In contrast, we set out a unified approach to max-FP and min-FP, and further develop a matrix extension based on the minorization-maximization (MM) theory in order to account for the ratios between matrices.

In the past studies of FP, Dinkelbach's algorithm [3] has long been the most popular method for the single-ratio FP. Although Dinkelbach's algorithm can be extended to the multi-ratio FP [4] with a max-min objective, its extension to the other multi-ratio cases including (1) and (2) remains an open problem. An extension attempt in [5] aims at the sum-of-ratios max-FP in (1), but a disproof of this result is given in [6] through a counterexample. Another work [7] shows that the min-FP in (2) amounts to minimizing the difference between two convex functions. Furthermore, [8] shows the NP-completeness of the sum-of-ratios problem (either max or min), and consequently many existing methods resort to branch and bound (B&B), e.g., [8]–[12] consider max-FP while [13]–[17] consider min-FP. In particular, the bounds used for B&B are not exchangeable between max-FP and min-FP. To avoid the exponential running time, other works suggest various heuristic algorithms, e.g., the "state space reduction" method [6] and the "harmony search" method [18].

Energy efficiency maximization for wireless links is one of the earliest applications of FP in the communication field. A survey may be found in [19]. In order to apply Dinkelbach's algorithm [3] or its generalized version [4], the energy efficiency problems in [19] are restricted to the single-link case and the max-min case. In comparison, the quadratic transform proposed in the more recent work [20] can tackle a much wider range of multi-ratio max-FP problems. Nevertheless, the use of the quadratic transform is still limited due to the gap between max-FP and min-FP. For instance, it does not work for the Cramér-Rao bounds (CRBs) minimization in a radar sensing task. The proposed unified approach can be viewed as a further generalization of the quadratic transform [20] that is capable of maximizing the signal-to-interference-plus-noise ratios (SINRs) and minimizing the CRBs. We illustrate the use of this new FP technique through two application cases: (i) joint radar sensing and (ii) multi-cell secure transmission.

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II. Optimization Technique

A. Max-FP Problems

We start with a brief review of the existing multi-ratio max-FP method called the quadratic transform.

Proposition 1 (Quadratic Transform [20]): The sum-of-ratios maximization problem in (1) is equivalent to

$$\underset{\boldsymbol{x}\in\mathcal{X},y_n\in\mathbb{R}}{\text{maximize}} \quad \sum_{n=1}^{N}\Big(2y_n\sqrt{A_n(\boldsymbol{x})}-y_n^2B_n(\boldsymbol{x})\Big) \tag{3}$$

in the sense that $\boldsymbol{x}^\star$ is a solution to (1) if and only if $(\boldsymbol{x}^\star, \{y_n^\star\})$ is a solution to (3), where $\{y_n\}$ is a set of auxiliary variables.

The benefit of transforming (1) into (3) is that the optimization over $\boldsymbol{x}$ (for fixed $\{y_n\}$) is a convex problem. Moreover, when $\boldsymbol{x}$ is held fixed, each y_n can be optimally updated as

$$y_n^\star = \frac{\sqrt{A_n(\boldsymbol{x})}}{B_n(\boldsymbol{x})}. \tag{4}$$

Thus, an alternating optimization between $\boldsymbol{x}$ and $\{y_n\}$ can be carried out efficiently in the new problem. It is shown in [20] that the alternating optimization yields a nondecreasing convergence of the objective value. Furthermore, if $A_n(\boldsymbol{x})$ and $B_n(\boldsymbol{x})$ are differentiable, then it attains a stationary point.

B. Min-FP Problems

We now consider the min-FP in (2). A natural idea is to reuse the above quadratic transform. One may rewrite (2) as

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \sum_{n=1}^{N}\frac{A_n(\boldsymbol{x})}{-B_n(\boldsymbol{x})} \tag{5}$$

in order to apply the max-FP technique. But this is problematic because (5) violates the positive-denominator requirement. Alternatively, one may flip every ratio and convert (2) to

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \sum_{n=1}^{N}\frac{B_n(\boldsymbol{x})}{A_n(\boldsymbol{x})}. \tag{6}$$

It can be shown that the above new problem boils down to using the harmonic mean to approximate the algebraic mean in (2). However, this approximation can be fairly loose.

The first main result of this paper is to show how the min-FP can be rewritten for alternating optimization.

Proposition 2 (Inverse Quadratic Transform): The sum-of-ratios minimization problem in (2) is equivalent to

$$\underset{\boldsymbol{x}\in\mathcal{X},\, y_n\in\mathbb{R}}{\text{minimize}} \quad \sum_{n=1}^{N}\frac{1}{2y_n\sqrt{B_n(\boldsymbol{x})}-y_n^2A_n(\boldsymbol{x})} \tag{7a}$$

$$\text{subject to} \quad 2y_n\sqrt{B_n(\boldsymbol{x})}-y_n^2A_n(\boldsymbol{x})>0,\ \forall n \tag{7b}$$

in the sense that $\boldsymbol{x}^\star$ is a solution to (2) if and only if $(\boldsymbol{x}^\star, \{y_n^\star\})$ is a solution to (7), where $\{y_n\}$ is a set of auxiliary variables.

Proof: When $\boldsymbol{x}$ is fixed, each y_n in (7) can be optimally determined as $y_n^\star = \sqrt{B_n(\boldsymbol{x})}/A_n(\boldsymbol{x})$. Substituting the above $y_n^\star$ into the new objective function (7a) recovers the original objective function in (2). Moreover, the new constraint (7b) can be satisfied automatically whenever $y_n^\star = \sqrt{B_n(\boldsymbol{x})}/A_n(\boldsymbol{x})$ regardless of the value of $\boldsymbol{x}$. The equivalence is then established. ■

Most importantly, for the new problem in (7), $\boldsymbol{x}$ and $\{y_n\}$ can be efficiently optimized in an alternating fashion. When y_n's are all fixed, notice that (7) is convex in $\boldsymbol{x}$. When $\boldsymbol{x}$ is fixed, y_n can be optimally determined as

$$y_n^\star = \frac{\sqrt{B_n(\boldsymbol{x})}}{A_n(\boldsymbol{x})}. \tag{8}$$

Unlike the max-FP case, we need the additional constraint (7b) for the min-FP reformulation. Otherwise, each y_n would tend to 0 from left and then the alternating optimization fails.

C. Mixed Max-and-Min FP Problems

We propose putting max-FP and min-FP in the same problem. Let each $f_n^+:\mathbb{R}\to\mathbb{R}$ be a concave increasing function for $1\le n\le N_0$, and let each $f_n^-:\mathbb{R}\to\mathbb{R}$ be a concave decreasing function for $N_0<n\le N$. The resulting *mixed max-and-min FP* problem is

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \ \sum_{n=1}^{N_0} f_n^+\left(\frac{A_n(\boldsymbol{x})}{B_n(\boldsymbol{x})}\right)+\sum_{n=N_0+1}^{N} f_n^-\left(\frac{A_n(\boldsymbol{x})}{B_n(\boldsymbol{x})}\right), \tag{9}$$

where the assumption of $A_n(\boldsymbol{x})$ and $B_n(\boldsymbol{x})$ inside each f_n^+ (resp. f_n^-) follows that of the max-FP (resp. min-FP). Intuitively, we wish to maximize all the ratios inside f_n^+ and minimize all the ratios inside f_n^-.

Proposition 3 (Unified Quadratic Transform): The mixed max-and-min FP problem in (9) is equivalent to

$$\begin{aligned}\underset{\boldsymbol{x}\in\mathcal{X},\, y_n\in\mathbb{R}}{\text{maximize}} \quad & \sum_{n=1}^{N_0} f_n^+\Big(2y_n\sqrt{A_n(\boldsymbol{x})}-y_n^2B_n(\boldsymbol{x})\Big)+\\ & \sum_{n=N_0+1}^{N} f_n^-\left(\frac{1}{2y_n\sqrt{B_n(\boldsymbol{x})}-y_n^2A_n(\boldsymbol{x})}\right)\end{aligned} \tag{10a}$$

$$\text{subject to} \quad 2y_n\sqrt{B_n(\boldsymbol{x})}-y_n^2A_n(\boldsymbol{x})>0,\ \forall n>N_0 \tag{10b}$$

in the sense that $\boldsymbol{x}^\star$ is a solution to (9) if and only if $(\boldsymbol{x}^\star, \{y_n^\star\})$ is a solution to (10), where $\{y_n\}$ is a set of auxiliary variables.

The above result can be readily verified by combining the proof of quadratic transform [20] and the proof of inverse quadratic transform from the last subsection. Again, an alternating optimization method can be carried out for the above new problem. When $\boldsymbol{x}$ is fixed, those y_n's inside f_n^+ are updated according to (4) and those y_n's inside f_n^- are updated according to (8); when y_n's are all fixed, $\boldsymbol{x}$ in (10) can be optimally updated via convex optimization. Next, we draw further insights into this unified method by the MM theory.

D. Connection to MM and Matrix Extension

It is worth looking at Proposition 3 from an MM point of view. Following [21], We denote by $g(\boldsymbol{x}|\hat{\boldsymbol{x}})$ the function $g:\mathcal{X}\to\mathbb{R}$ parameterized by $\hat{\boldsymbol{x}}$ with the input $\boldsymbol{x}\in\mathcal{X}$.

The optimal updates of $y_n^\star$ in (4) and (8) are treated as two functions of $\boldsymbol{x}$:

$$Y_n^+(\boldsymbol{x})=\frac{\sqrt{A_n(\boldsymbol{x})}}{B_n(\boldsymbol{x})} \quad \text{and} \quad Y_n^-(\boldsymbol{x})=\frac{\sqrt{B_n(\boldsymbol{x})}}{A_n(\boldsymbol{x})}. \tag{11}$$

We then define $g(\boldsymbol{x}|\hat{\boldsymbol{x}})$ in our case to be

$$g(\boldsymbol{x}|\hat{\boldsymbol{x}}) = \sum_{n\le N_0} f_n^+ \left(2Y_n^+(\hat{\boldsymbol{x}})\sqrt{A_n(\boldsymbol{x})} - (Y_n^+(\hat{\boldsymbol{x}}))^2 B_n(\boldsymbol{x})\right) + \sum_{n>N_0} f_n^- \left(\frac{1}{2Y_n^-(\hat{\boldsymbol{x}})\sqrt{B_n(\boldsymbol{x})} - (Y_n^-(\hat{\boldsymbol{x}}))^2 A_n(\boldsymbol{x})}\right) \quad (12)$$

Some algebra shows that

$$g(\hat{\boldsymbol{x}}|\hat{\boldsymbol{x}}) = \sum_{n\le N_0} f_n^+ \left(\frac{A_n(\hat{\boldsymbol{x}})}{B_n(\hat{\boldsymbol{x}})}\right) + \sum_{n>N_0} f_n^- \left(\frac{A_n(\hat{\boldsymbol{x}})}{B_n(\hat{\boldsymbol{x}})}\right),$$
$$g(\boldsymbol{x}|\hat{\boldsymbol{x}}) = g(\boldsymbol{x}|\boldsymbol{x}).$$

Thus, $g(\boldsymbol{x}|\hat{\boldsymbol{x}})$ is a *surrogate function* [22] with respect to the mixed max-and-min objective in (9). In light of the MM theory [22], we immediately have the following result.

Proposition 4 (Convergence Analysis): The alternating optimization between $\boldsymbol{x}$ and $\{y_n\}$ in (10) yields a nondecreasing convergence of the original mixed max-and-min FP objective in (9). Furthermore, if all the functions $\{f_n^+, f_n^-, A_n, B_n\}$ are differentiable, then the alternating optimization converges to a stationary point of the original problem (9).

Moreover, by generalizing the surrogate function (12) for the matrix case, we obtain a matrix extension of Proposition 3. Now assume that $\boldsymbol{A}_n(\boldsymbol{x}) \in \mathbb{H}_+^{d\times d}$ is positive semi-definite and $\boldsymbol{B}_n(\boldsymbol{x}) \in \mathbb{H}_{++}^{d\times d}$ is positive definite. We denote by $\sqrt{\boldsymbol{M}} \in \mathbb{C}^{d\times \ell}$ the square root of $\boldsymbol{M}$ for some $\ell \le d$, i.e., $\sqrt{\boldsymbol{M}}(\sqrt{\boldsymbol{M}})^H = \boldsymbol{M}$. Let $f_n^+ : \mathbb{H}_+^{\ell\times\ell} \to \mathbb{R}$ be a concave increasing function so that $f_n^+(\boldsymbol{M}) > f_n^+(\boldsymbol{M}')$ if $\boldsymbol{M} \succ \boldsymbol{M}'$. Likewise, let $f_n^- : \mathbb{H}_+^{\ell\times\ell} \to \mathbb{R}$ be a concave decreasing function so that $f_n^-(\boldsymbol{M}) < f_n^-(\boldsymbol{M}')$ if $\boldsymbol{M} \succ \boldsymbol{M}'$. Further, assume that $f_n^-((\sqrt{\boldsymbol{A}}^H \boldsymbol{B}^{-1}\sqrt{\boldsymbol{A}})^{-1}) = f_n^-(\sqrt{\boldsymbol{B}}^H \boldsymbol{A}^{-1}\sqrt{\boldsymbol{B}})$. We then state the matrix extension of the unified quadratic transform in the following proposition without proof.

Proposition 5 (Matrix Extension): The mixed max-and-min matrix FP problem

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \sum_{n\le N_0} f_n^+\left(\sqrt{\boldsymbol{A}}_n^H(\boldsymbol{x})\boldsymbol{B}_n^{-1}(\boldsymbol{x})\sqrt{\boldsymbol{A}}_n(\boldsymbol{x})\right) + \sum_{n>N_0} f_n^-\left(\sqrt{\boldsymbol{A}}_n^H(\boldsymbol{x})\boldsymbol{B}_n^{-1}(\boldsymbol{x})\sqrt{\boldsymbol{A}}_n(\boldsymbol{x})\right), \quad (13)$$

is equivalent to

$$\underset{\boldsymbol{x}\in\mathcal{X},\,\boldsymbol{Y}_n\in\mathbb{C}^{d\times\ell}}{\text{maximize}} \quad \sum_{n\le N_0} f_n^+\left(\boldsymbol{Q}_n^+\right) + \sum_{n>N_0} f_n^-\left((\boldsymbol{Q}_n^-)^{-1}\right) \quad (14a)$$
$$\text{subject to} \quad \boldsymbol{Q}_n^- \succ \boldsymbol{0},\ \forall n > N_0, \quad (14b)$$

with the shorthand

$$\boldsymbol{Q}_n^+ = \sqrt{\boldsymbol{A}}_n^H(\boldsymbol{x})\boldsymbol{Y}_n + \boldsymbol{Y}_n^H\sqrt{\boldsymbol{A}}_n(\boldsymbol{x}) - \boldsymbol{Y}_n^H\boldsymbol{B}_n(\boldsymbol{x})\boldsymbol{Y}_n, \quad (15)$$
$$\boldsymbol{Q}_n^- = \sqrt{\boldsymbol{B}}_n^H(\boldsymbol{x})\boldsymbol{Y}_n + \boldsymbol{Y}_n^H\sqrt{\boldsymbol{B}}_n(\boldsymbol{x}) - \boldsymbol{Y}_n^H\boldsymbol{A}_n(\boldsymbol{x})\boldsymbol{Y}_n, \quad (16)$$

where $\{\boldsymbol{Y}_n \in \mathbb{C}^{d\times\ell}\}$ is a set of auxiliary matrix variables. For the new problem in (14), $\boldsymbol{x}$ and $\{\boldsymbol{Y}_n\}$ can be efficiently optimized in an alternating fashion. When $\boldsymbol{x}$ is fixed, those $\boldsymbol{Y}_n$'s inside $\boldsymbol{Q}_n^+$ are updated as $\boldsymbol{Y}_n = \boldsymbol{B}_n^{-1}(\boldsymbol{x})\sqrt{\boldsymbol{A}_n}(\boldsymbol{x})$ and those $\boldsymbol{Y}_n$'s inside $\boldsymbol{Q}_n^{-1}$ are updated as $\boldsymbol{Y}_n = \boldsymbol{A}_n^{-1}(\boldsymbol{x})\sqrt{\boldsymbol{B}_n}(\boldsymbol{x})$; when $\boldsymbol{Y}_n$'s are all fixed, $\boldsymbol{x}$ in (14) can be optimally updated via convex optimization.

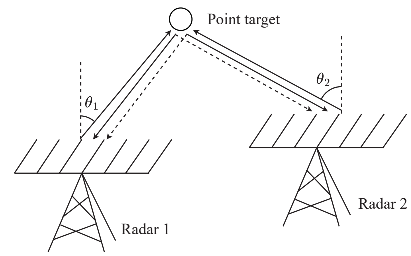

Fig. 1. Two radar sets detect a common point target in the same spectrum band. The dashed lines represent the interference between them.

III. Application Cases

A. Joint Radar Sensing

Consider M radar sets that work on the same spectrum band to detect a common point target. Assume that the radar set m has N_m^T transmit antennas and N_m^R receive antennas, both arranged as a half-wavelength spaced uniform linear array (ULA). Let L be the number of samples taken of the echo signal at each radar set. For each radar set m, denote by θ_m the direction of arrival (DOA) of the target as illustrated in Fig. 1, $\boldsymbol{S}_m \in \mathbb{C}^{N_m^T\times L}$ the transmit waveform matrix, $\boldsymbol{a}_m^T(\theta_m) \in \mathbb{C}^{N_m^T\times 1}$ the steering vector of the transmit antennas, $\mathbf{a}_m^R(\theta_m) \in \mathbb{C}^{N_m^R\times 1}$ the steering vector of the receive antennas, and $\boldsymbol{Z}_m \sim \mathcal{CN}(\boldsymbol{0}, \sigma_m^2 \boldsymbol{I}_{N_m^R})$ the background noise. Moreover, for a pair of radar sets m and m', let $\beta_{mm'}$ be the reflection coefficient from radar m' to radar m (which depends on the pathloss and the radar cross section). Thus, the sampled echo signal $\boldsymbol{F}_m \in \mathbb{C}^{N_m^R\times L}$ at the radar set m is given by

$$\boldsymbol{F}_m = \sum_{m'=1}^{M} \beta_{mm'}\boldsymbol{a}_m^R(\theta_m)(\boldsymbol{a}_{m'}^T(\theta_{m'}))^H\boldsymbol{S}_{m'} + \boldsymbol{Z}_m. \quad (17)$$

Each radar set m aims to estimate the DoA θ_m based on $\boldsymbol{F}_m$. We seek the optimal waveform design of each $\boldsymbol{S}_m$ under the power constraint P_m, i.e., with the squared Frobenius norm $\|\boldsymbol{S}_m\|_F^2 \le P_m$, in order to minimize the sum mean squared error (MSE) of all the θ_m estimates.

But the MSE of each θ_m is difficult to evaluate directly. A common practice is to use the CRB to approximate the actual MSE, as discussed in the sequel. Define the response matrix

$$\boldsymbol{G}_{mm'} = \beta_{mm'}\boldsymbol{a}_m^R(\theta_m)(\boldsymbol{a}_{m'}^T(\theta_{m'}))^H. \quad (18)$$

Then the vectorization $\boldsymbol{f}_m = \mathrm{vec}(\boldsymbol{F}_m)$ can be computed as

$$\boldsymbol{f}_m = (\boldsymbol{I}_L\otimes\boldsymbol{G}_{mm})\boldsymbol{s}_m + \sum_{m'\ne m}(\boldsymbol{I}_L\otimes\boldsymbol{G}_{mm'})\boldsymbol{s}_{m'} + \boldsymbol{z}_m, \quad (19)$$

where $\boldsymbol{s}_m = \mathrm{vec}(\boldsymbol{S}_m)$, and $\boldsymbol{z}_m = \mathrm{vec}(\boldsymbol{Z}_m)$. Letting $\hat{\boldsymbol{f}}_m = \boldsymbol{f}_m - (\boldsymbol{I}_L\otimes\boldsymbol{G}_{mm})\boldsymbol{s}_m$, compute $\boldsymbol{K}_m = \mathbb{E}[\hat{\boldsymbol{f}}_m\hat{\boldsymbol{f}}_m^H]$ as

$$\boldsymbol{K}_m = \sum_{m'\ne m}\left(\boldsymbol{I}_L\otimes\boldsymbol{G}_{mm'}\right)\boldsymbol{s}_{m'}\boldsymbol{s}_{m'}^H\left(\boldsymbol{I}_L\otimes\boldsymbol{G}_{mm'}\right)^H + \sigma_m^2\boldsymbol{I}_{LN_m^R}. \quad (20)$$

According to [23], the Fisher information about θ_m is

$$J_m(\theta_m) = 2\boldsymbol{s}_m^H(\boldsymbol{I}_L \otimes \dot{\boldsymbol{G}}_{mm})^H \boldsymbol{K}_m^{-1}(\boldsymbol{I}_L \otimes \dot{\boldsymbol{G}}_{mm})\boldsymbol{s}_m, \quad (21)$$

where $\dot{\boldsymbol{G}}_{mm}$ is the partial derivative of $\boldsymbol{G}_{mm}$ with respect to θ_m. Since the CRB amounts to the reciprocal of the Fisher information, the sum-of-MSEs problem can be approximated as the following sum-of-CRBs problem:

$$\underset{\boldsymbol{s}_m}{\text{minimize}} \quad \sum_{m=1}^{M} \frac{1}{J_m(\theta_m)} \quad (22a)$$

$$\text{subject to} \quad \|\boldsymbol{s}_m\|^2 \le P_m, \forall m. \quad (22b)$$

In order to apply the proposed unified approach, one may rewrite (22) in the standard form as in Proposition 5:

$$\underset{\boldsymbol{s}_m}{\text{maximize}} \quad \sum_{m=1}^{M} f_m^+\left(\sqrt{\boldsymbol{V}}_m^H \boldsymbol{K}_m^{-1}\sqrt{\boldsymbol{V}}_m\right) \quad (23a)$$

$$\text{subject to} \quad \|\boldsymbol{s}_m\|^2 \le P_m, \quad \forall m, \quad (23b)$$

where $\sqrt{\boldsymbol{V}}_m = \left(\boldsymbol{I}_L \otimes \dot{\boldsymbol{G}}_{mm}\right)\boldsymbol{s}_m$ and $f_m^+(c) = -1/(2c)$.

However, the matrix denominator $\boldsymbol{K}_m$ is not convex in $\boldsymbol{s}_m$, so the optimization of $\boldsymbol{s}_m$ is still difficult even after applying the unified quadratic transform. We propose addressing this issue by the Schur complement. As a result, (23) is recast to

$$\underset{\boldsymbol{s}_m,\boldsymbol{U}_m}{\text{maximize}} \quad \sum_{m=1}^{M} f_m^+\left(\sqrt{\boldsymbol{V}}_m^H \boldsymbol{\Lambda}_m^{-1}\sqrt{\boldsymbol{V}}_m\right) \quad (24a)$$

$$\text{subject to} \quad \|\boldsymbol{s}_m\|^2 \le P_m, \forall m \quad (24b)$$

$$\boldsymbol{U}_m \in \mathbb{C}^{LN_m^T \times LN_m^T}, \forall m \quad (24c)$$

$$\begin{bmatrix} \boldsymbol{U}_m & \boldsymbol{s}_m \\ \boldsymbol{s}_m^H & 1 \end{bmatrix} \succeq \boldsymbol{0}, \forall m, \quad (24d)$$

where $\boldsymbol{\Lambda}_m = \sum_{m' \neq m}(\boldsymbol{I}_L \otimes \boldsymbol{G}_{mm'})\boldsymbol{U}_{m'}(\boldsymbol{I}_L \otimes \boldsymbol{G}_{mm'})^H + \sigma_m^2 \boldsymbol{I}$. Applying the unified quadratic transform to the above problem yields the following reformulation:

$$\underset{\boldsymbol{s}_m,\boldsymbol{U}_m,\boldsymbol{Y}_m}{\text{maximize}} \quad \sum_{m=1}^{M} f_m^+\left(Q_m^+\right) \quad (25a)$$

$$\text{subject to} \quad \text{(24b)–(24d)} \quad (25b)$$

$$\boldsymbol{Y}_m \in \mathbb{C}^{LN_m^R}, \forall m, \quad (25c)$$

where $Q_m^+ = \sqrt{\boldsymbol{V}}_m^H \boldsymbol{Y}_m + \boldsymbol{Y}_m^H \sqrt{\boldsymbol{V}}_m - \boldsymbol{Y}_m^H \boldsymbol{\Lambda}_m \boldsymbol{Y}_m$. Observe that the above problem is jointly convex in $\{\boldsymbol{s}_m\}$ and $\{\boldsymbol{U}_m\}$ when $\{\boldsymbol{Y}_m\}$ is fixed, while each $\boldsymbol{Y}_m$ is iteratively updated as $\boldsymbol{Y}_m^\star = \boldsymbol{\Lambda}_m^{-1}\sqrt{\boldsymbol{V}}_m$.

The simulation setting follows. Consider 5 radar sets with $L = 4$. For $m = 1, 2, \ldots, 5$, let N_m^T in order be $(4, 2, 2, 2, 2)$, let N_m^R in order be $(6, 4, 4, 4, 4)$, and let θ_m in order be $(\frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{2}{5}\pi, \frac{3}{7}\pi)$. Let every $\beta_{mm'} = 1$ and every $P_m =$ 30 dBm. We compare the proposed method with separate successive convex approximation in [24]. As shown in Fig. 2, the benchmark algorithm converges faster than the proposed algorithm, and yet the latter ultimately achieves a much lower sum of the CRBs. The convergence of the proposed method is around 70% lower than the starting point and is around 25% lower than that of the existing method in [24].

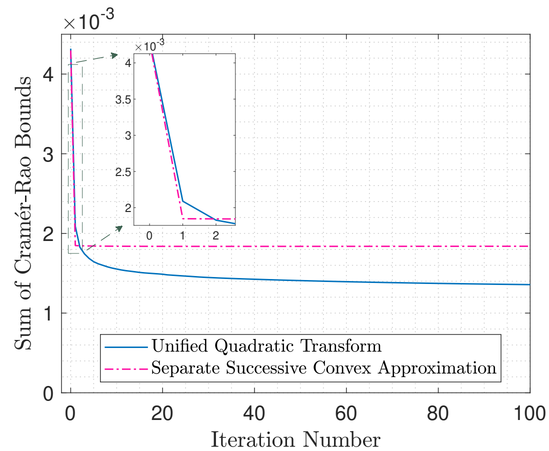

Fig. 2. Minimizing sum of CRBs across 5 radar sets.

B. Secure Transmission

Consider L base-stations (BSs) each serving a legitimate downlink user terminal. Assume that the first K ($K \le L$) BSs face one eavesdropper each. We use p_i to denote the transmit power of BS i, $h_{ji} \in \mathbb{C}$ the channel from BS i to legitimate user j, $\tilde{h}_{ki} \in \mathbb{C}$ the channel from BS i to eavesdropper k, σ_i^2 the background noise power at legitimate user i, and $\tilde{\sigma}_k^2$ the background noise power at eavesdropper k. Assuming that cross-cell interference is treated as noise, the achievable secure data rate of BS $i = 1, \ldots, K$ with eavesdropper is given by

$$R_i = \log\left(1 + \frac{|h_{ii}|^2 p_i}{\sum_{j \neq i}|h_{ij}|^2 p_j + \sigma_i^2}\right) - \log\left(1 + \frac{|\tilde{h}_{ii}|^2 p_i}{\sum_{j \neq i}|\tilde{h}_{ij}|^2 p_j + \tilde{\sigma}_i^2}\right), \quad (26)$$

while the data rate of BS $i = K+1, \ldots, L$ is given by

$$R_i = \log\left(1 + \frac{|h_{ii}|^2 p_i}{\sum_{j \neq i}|h_{ij}|^2 p_j + \sigma_i^2}\right). \quad (27)$$

We seek the optimal powers $(p_1, \ldots, p_L)$ that maximize the sum rates under the power constraint P, i.e.,

$$\underset{p_i}{\text{maximize}} \quad \sum_{i=1}^{L} R_i \quad (28a)$$

$$\text{subject to} \quad 0 \le p_i \le P, \text{ for } i = 1, \ldots, L. \quad (28b)$$

We remark that the above problem is a mixed max-and-min FP problem, so the existing quadratic transform in [20] does not work for it. But the proposed unified quadratic transform is applicable here. First, rewrite the objective function as

$$\sum_{i=1}^{L} R_i = \sum_{i=1}^{L}\log\left(1 + \frac{|h_{ii}|^2 p_i}{\sum_{j=1, j\neq i}^{L}|h_{ij}|^2 p_j + \sigma_i^2}\right) + \sum_{k=1}^{K}\log\left(1 - \frac{|\tilde{h}_{kk}|^2 p_k}{\sum_{j=1}^{L}|\tilde{h}_{kj}|^2 p_j + \tilde{\sigma}_k^2}\right). \quad (29)$$

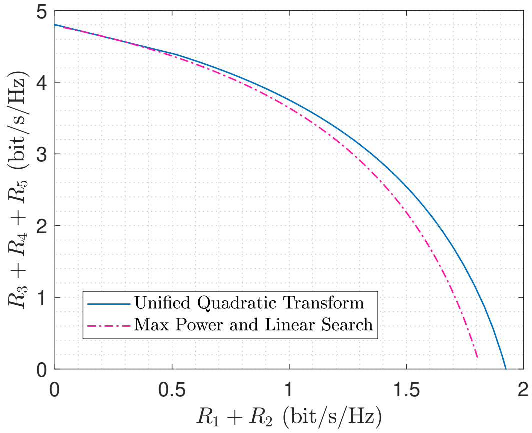

Fig. 3. Tradeoff between the sum-rates $R_1 + R_2$ at risk of eavesdropping and the sum-rates $R_3 + R_4 + R_5$ free of eavesdropping.

We now treat each $\frac{|h_{ii}|^2 p_i}{\sum_{j=1,j\neq i}^{L} |h_{ij}|^2 p_j + \sigma_i^2}$ as the ratio nested in the concave increasing function $f_i^+(r) = \log(1+r)$, for $i = 1,\ldots,L$, and treat each $\frac{|\tilde{h}_{kk}|^2 p_k}{\sum_{j=1}^{L} |\tilde{h}_{kj}|^2 p_j + \tilde{\sigma}_k^2}$ as the ratio nested in the concave decreasing function $f_k^-(r) = \log(1-r)$, for $k = 1,\ldots,K$. Thus, the unified quadratic transform in Proposition 3 immediately applies. The resulting new problem is

$$\underset{p_i, y_i, \tilde{y}_k}{\text{maximize}} \quad \sum_{i=1}^{L} \log(1 + Q_i^+) + \sum_{k=1}^{K} \log\left(1 - \frac{1}{Q_k^-}\right) \tag{30a}$$

$$\text{subject to} \quad 0 \le p_i \le P, \text{ for } i = 1,\ldots,L \tag{30b}$$

$$Q_k^- > 0, \text{ for } k = 1,\ldots,K, \tag{30c}$$

where $Q_i^+ = 2y_i\sqrt{|h_{ii}|^2 p_i} - y_i^2\left(\sum_{j\neq i} |h_{ij}|^2 p_j + \sigma_i^2\right)$ and $Q_k^- = 2\tilde{y}_k\sqrt{\sum_j |\tilde{h}_{kj}|^2 p_j + \tilde{\sigma}_k^2} - \tilde{y}_k^2 |\tilde{h}_{kk}|^2 p_k$.

Again, in an iterative fashion, $\{p_i\}$ can be efficiently updated via convex optimization when $\{y_i, \tilde{y}_k\}$ are held fixed, and then $\{y_i, \tilde{y}_k\}$ are updated according to (4) and (8).

For the simulation, we set $L = 5$, $K = 2$, $\sigma_i^2 = -90$ dBm, $\tilde{\sigma}_k^2 = -80$ dBm, and $P = 10$ dBm. Let $|h_{ii}|^2 = 1, 0.95, 0.85, 0.90, 0.94$ for $i = 1,\ldots,5$, let $|\tilde{h}_{kk}|^2 = 0.50, 0.48$ for $k = 1, 2$, and let $|h_{ji}|^2 = |\tilde{h}_{ki}|^2 = 0.1$ for any $i \neq j$ and $i \neq k$. We plot the tradeoff between the sum secure rates $R_1 + R_2$ (with eavesdropping) and the sum rates $R_3 + R_4 + R_5$ (without eavesdropping). We fix w_i to be 1 for $i \in \{1, 2\}$, and set the rest w_i's for $i \in \{3, 4, 5\}$ to the same value η ranging from 0.001 to 100. We consider the following benchmark called *max power and linear search*: let $p_1 = p_2 = \rho$ and let $p_3 = p_4 = p_5 = \rho'$; fix either ρ or ρ' to the peak power and tune the other by linear search. The figure shows that the proposed method outperforms the benchmark significantly. For instance, if $R_1 + R_2$ is fixed at 1.5 bits/s/Hz, the FP-based method enhances $R_3 + R_4 + R_5$ by around 13% as compared to the benchmark method.

IV. Conclusion

Differing most previous studies that consider min-FP and max-FP separately, this work proposes a unified approach to these two types of FP. The proposed method is closely related to MM and can be further extended to the matrix ratios. In addition, we illustrate the use of this new FP technique in target sensing and secure transmission.

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