INVERSE QUADRATIC TRANSFORM FOR MINIMIZING A SUM OF RATIOS

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ABSTRACT

A major challenge with the multi-ratio Fractional Program (FP) is that the existing methods for the maximization problem typically do not work for the minimization case. We propose a novel technique called inverse quadratic transform for the sum-of-ratios minimization problem. Its main idea is to reformulate the min-FP problem in a form amenable to efficient iterative optimization. Furthermore, this transform can be readily extended to a general cost-function-of-multiple-ratios minimization problem. We also give a Majorization-Minimization (MM) interpretation of the inverse quadratic transform, showing that all those desirable properties of MM can be carried over to the new technique. Moreover, we demonstrate the application of inverse quadratic transform in minimizing the Age-of-Information (AoI) of data networks.

1. INTRODUCTION

Fractional Program (FP), namely the problem of optimizing one or multiple ratios, has long been recognized as a fundamental one in extensive areas ranging from portfolio to management science, data science, and wireless communications [1, 2]. This paper focuses on the minimization case of FP. The main results are two-fold: a new method for minimizing a sum of ratios (or, more generally, a cost function of multiple ratios), and its application to Age-of-Information (AoI).

Actually, the line between maximization and minimization is rather vague in the early studies of FP—which mostly consider only one ratio, since the minimization problem can be immediately converted to the maximization by flipping the ratio. As a result, the classical methods of FP, i.e., Charnes-Cooper algorithm [3,4] and Dinkelbach's algorithm [5], work for both maximization and minimization of a single ratio. As an extension to the multiple-ratios FP, we can convert a min-max ratios problem to a max-min ratios problem by flipping every ratio term, and then a generalized Dinkelbach's algorithm [6] applies. Nevertheless, the conversion from minimization to maximization, or the other way around, is difficult for the sum-of-ratios case, so maximization and minimization are often dealt with separately in the literature.

Because the sum-of-ratios problem is NP-complete [7], many existing methods build upon the branch-and-bound paradigm, e.g., [7–11] for the minimization case while [12–16] for the maximization case. These methods all guarantee the global optimum but are not scalable due to the exponential time complexities. The authors of [17] suggest an extended Dinkelbach's algorithm for the sum-of-ratios maximization, but its effectiveness is disproved in [18] through a counterexample. The work in [18] proposes casting the sum-of-ratios max problem into a "state space" wherein each ratio term corresponds to a state variable; the main idea is to gradually reduce the state space according to the variable feasibility. However, as pointed out in [1], the performance of the above method is not provable. The harmony search method in [19] is another heuristic approach to the sum-of-ratios max problem. Moreover, inspired by the classical Dinkelbach's method, [20] rewrites the sum-of-ratios minimization as the difference minimization between two convex functions, which is a well-known nonconvex optimization problem. A more recent work [2] proposes a quadratic transformation of the multiple-ratios max-FP whereby an efficient iterative optimization can be performed. The present paper aims to extend the quadratic transform of [2] to the min-FP.

The second part of the paper concerns with a novel application of FP in the rate control of a multiple-sources network in order to minimize the overall AoI. The notion of AoI was introduced in the early 2010s [21] to quantify the freshness of data packets in networks, which refers to the time elapsed since the generation of the last successfully received update information about the source node. The pioneer work [21] proposes a descent algorithm for minimizing the AoI of a single source. Considering multiple sources under a homogeneous setup (i.e., all the sources have the same service rate), [22, 23] show that the rate control problem can be optimally solved in closed form. Rate control is much more challenging in terms of optimization for the heterogeneous setup. The authors of [24] suggest a quasilinear approximation of the two-source heterogenous case to facilitate rate control. Some more recent works consider the AoI minimization task in various application scenarios, e.g., [25] proposes a reinforcement

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algorithm for the sensor networks, and [26] proposes an ant-colony heuristic algorithm for the Unmanned Aerial Vehicle (UAV)-assisted wireless networks. In contrast, this work focuses on the optimization aspect of the AoI problem. To the best of our knowledge, this is the very first work that views AoI from an FP perspective.

2. SUM-OF-RATIOS MINIMIZATION PROBLEM

Consider N ratios. Each ratio consists of the numerator function $A_n(\boldsymbol{x})$ and the denominator function $B_n(\boldsymbol{x})$ of the variable $\boldsymbol{x} \in \mathbb{R}^d$, $n = 1, \ldots, N$. Assume that each $A_n(\boldsymbol{x}) > 0$ is a positive convex function while each $B_n(\boldsymbol{x}) > 0$ is a positive concave function[1]. With a nonempty convex constraint set $\mathcal{X} \subseteq \mathbb{R}^d$, the *sum-of-ratios minimization* problem is

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \sum_{n=1}^{N} \frac{A_n(\boldsymbol{x})}{B_n(\boldsymbol{x})}. \tag{1}$$

The above problem is nonconvex in general despite the convexity of $A_n(\boldsymbol{x})$ and the concavity of $B_n(\boldsymbol{x})$. An extension is to minimize a cost function of ratios. Now let the ratio vector $\left(\frac{A_1}{B_1}, \ldots, \frac{A_N}{B_N}\right)$ be the input of the cost function $U : \mathbb{R}^N \to \mathbb{R}$, which, by convention, is assumed to be monotonically increasing with each $\frac{A_n}{B_n}$ and is jointly convex in $\left(\frac{A_1}{B_1}, \ldots, \frac{A_N}{B_N}\right)$. The *cost-function-of-multiple-ratios minimization* problem is

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad U\left(\frac{A_1(\boldsymbol{x})}{B_1(\boldsymbol{x})}, \ldots, \frac{A_N(\boldsymbol{x})}{B_N(\boldsymbol{x})}\right). \tag{2}$$

Clearly, the above problem encompasses the sum-of-ratios minimization problem as a special case.

3. PROPOSED METHOD

3.1. *Two Common Misconceptions*

It is intriguing to rewrite (1) as

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \sum_{n=1}^{N} \frac{A_n(\boldsymbol{x})}{-B_n(\boldsymbol{x})} \tag{3}$$

so as to apply the existing method in [2] for the sum-of-ratios maximization problem. But this is problematic since max-FP requires both numerator and denominator to be positive.

Another possible approach is to rewrite (1) as

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \sum_{n=1}^{N} \frac{B_n(\boldsymbol{x})}{A_n(\boldsymbol{x})} \tag{4}$$

by flipping all the ratios. The two problems, (1) and (4), are not equivalent in general unless $N = 1$. It turns out that solving (4) can be interpreted as minimizing a lower bound on the sum-of-ratios objective. To see this, just treat $\frac{1}{N}\sum_{n=1}^{N}\frac{A_n}{B_n}$ as an arithmetic mean and then bound it below by the harmonic mean $N(\sum_n \frac{B_n}{A_n})^{-1}$; now recognize the problem in (4) as minimizing the harmonic mean. The above approach may yield quite poor performance because the harmonic mean can be a loose bound to the arithmetic mean.

In contrast, the proposed method can be interpreted as using a novel upper bound to approximate the sum-of-ratios objective. Furthermore, as discussed in Section 3.3, this approximation can be justified from a Majorization-Minimization (MM) point of view.

3.2. *Inverse Quadratic Transform*

We now present the main result of this work:

Proposition 1. *The sum-of-ratios minimization problem in* (1) *is equivalent to*

$$\underset{\boldsymbol{x}\in\mathcal{X},\,\boldsymbol{y}\in\mathbb{R}^N}{\textit{minimize}} \quad \sum_{n=1}^{N} \frac{1}{2y_n\sqrt{B_n(\boldsymbol{x})} - y_n^2 A_n(\boldsymbol{x})} \tag{5a}$$

$$\textit{subject to} \quad 2y_n\sqrt{B_n(\boldsymbol{x})} - y_n^2 A_n(\boldsymbol{x}) > 0,\ \forall n \tag{5b}$$

in the sense that $\boldsymbol{x}^\star$ *is a solution to* (1) *if and only if* $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ *is a solution to* (5).

Proof. When $\boldsymbol{x}$ is held fixed, the auxiliary variable $\boldsymbol{y}$ in (5) can be optimally determined as $y_n^\star = \sqrt{B_n(\boldsymbol{x})}/A_n(\boldsymbol{x})$. With each $y_n^\star$ substituted into the new objective (5a) as a function of $\boldsymbol{x}$, we recover the original sum-of-ratios objective in (1). In particular, the new constraint (5b) is satisfied automatically under $y_n^\star = \sqrt{B_n(\boldsymbol{x})}/A_n(\boldsymbol{x})$ regardless of the value of $\boldsymbol{x}$. The equivalence is then verified. □

Differing from the quadratic transform in [2] for the max-FP, the inverse quadratic transform in Proposition 1 for the min-FP needs to introduce an additional constraint (5b). This subtle difference is somewhat unexpected. We remark that (5b) is critical to the optimal update $y_n^\star = \sqrt{B_n(\boldsymbol{x})}/A_n(\boldsymbol{x})$ in (5) for fixed $\boldsymbol{x}$; otherwise we would have let $y_n^\star \to 0^-$ and consequently the new objective would go off to infinity.

Furthermore, the inverse quadratic transform can be extended to the cost-function-of-multiple-ratios minimization problem, as stated in the following corollary.

Corollary 1. *The cost-function-of-multiple-ratios minimization problem in* (2) *is equivalent to*

$$\underset{\boldsymbol{x}\in\mathcal{X},\,\boldsymbol{y}\in\mathbb{R}^N}{\textit{minimize}} \quad U\left(\frac{1}{2y_1\sqrt{B_1(\boldsymbol{x})} - y_1^2 A_1(\boldsymbol{x})}, \ldots, \frac{1}{2y_N\sqrt{B_N(\boldsymbol{x})} - y_N^2 A_N(\boldsymbol{x})}\right) \tag{6a}$$

$$\textit{subject to} \quad 2y_n\sqrt{B_n(\boldsymbol{x})} - y_n^2 A_n(\boldsymbol{x}) > 0,\ \forall n \tag{6b}$$

in the sense that $\boldsymbol{x}^\star$ *is a solution to* (2) *if and only if* $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ *is a solution to* (6).

[1] By symmetry, max-FP works often assume concave A_n and convex B_n.

Algorithm 1 Iterative Algorithm for Convex-Concave FP

1: Initialize $\boldsymbol{x}$ to some feasible point in $\mathcal{X}$
2: **repeat**
3: Update each auxiliary variable $y_n = \sqrt{B_n(\boldsymbol{x})}/A_n(\boldsymbol{x})$
4: Solve the convex problem of $\boldsymbol{x}$ in (6) for fixed $\boldsymbol{y}$
5: **until** the objective value converges

Our discussion in the rest of the section focuses on the above general form of the inverse quadratic transform. We propose optimizing $\boldsymbol{x}$ and $\boldsymbol{y}$ alternatingly in the new problem (6). When $\boldsymbol{x}$ is held fixed, each y_n is optimally determined as $y_n^\star = \frac{\sqrt{B_n(\boldsymbol{x})}}{A_n(\boldsymbol{x})}$. Further, according to the composition rule that preserves convexity, optimizing $\boldsymbol{x}$ in (6) under fixed $\boldsymbol{y}$ turns out to be a convex problem, so the iterative update on $\boldsymbol{x}$ can be performed efficiently by the standard optimization. Algorithm 1 summarizes the above optimization procedure based on inverse quadratic transform.

It can be easily seen that the new objective in (6a) is monotonically decreasing by the alternating optimization between $\boldsymbol{x}$ and $\boldsymbol{y}$. More importantly, we will show in the next subsection that the original cost-function-of-multiple-ratios objective in (2) has a monotonically decreasing convergence to a stationary point under certain conditions.

3.3. *Connection to Majorization-Minimization*

We now connect Algorithm 1 to the MM theory by showing that the inverse quadratic transform in Corollary 1 (or its special case as stated in Proposition 1) can be interpreted as constructing a *surrogate function* for the min-FP. To ease notation, we write the original cost-function-of-multiple-ratios objective as

$$f(\boldsymbol{x}) = U\left(\frac{A_1(\boldsymbol{x})}{B_1(\boldsymbol{x})}, \ldots, \frac{A_N(\boldsymbol{x})}{B_N(\boldsymbol{x})}\right). \tag{7}$$

The updating formula for y_n in step 3 of Algorithm 1 is denoted by

$$p_n(\boldsymbol{x}) = \frac{\sqrt{A_n(\boldsymbol{x})}}{B_n(\boldsymbol{x})}. \tag{8}$$

Suppose $\hat{\boldsymbol{x}}$ is the latest update of $\boldsymbol{x}$ in Algorithm 1, and thus each y_n is now updated to $p_n(\hat{\boldsymbol{x}})$. Substituting $y_n = p_n(\hat{\boldsymbol{x}})$ in the new objective (6) gives rise to a function of $\boldsymbol{x}$ conditioned on $\hat{\boldsymbol{x}}$:

$$\begin{aligned} g(\boldsymbol{x}|\hat{\boldsymbol{x}}) = U\Bigg(& \frac{1}{2p_1(\hat{\boldsymbol{x}})\sqrt{B_1(\boldsymbol{x})} - p_1^2(\hat{\boldsymbol{x}})A_1(\boldsymbol{x})}, \\ & \ldots, \frac{1}{2p_N(\hat{\boldsymbol{x}})\sqrt{B_N(\boldsymbol{x})} - p_N^2(\hat{\boldsymbol{x}})A_N(\boldsymbol{x})}\Bigg). \end{aligned} \tag{9}$$

It can be readily shown that

$$g(\hat{\boldsymbol{x}}|\hat{\boldsymbol{x}}) = f(\hat{\boldsymbol{x}}). \tag{10}$$

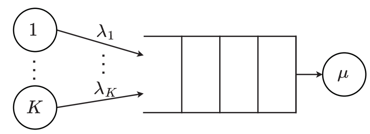


Fig. 1. An AoI model with K source nodes and a server.

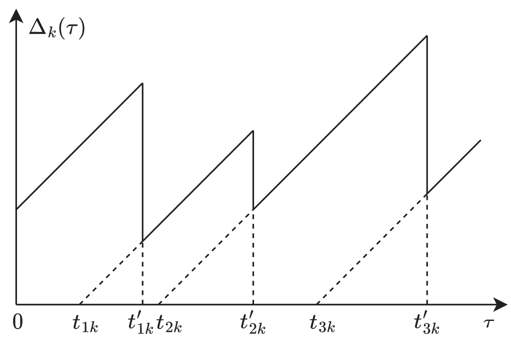


Fig. 2. A typical curve of AoI versus time.

Moreover, for arbitrary $(\boldsymbol{x}, \hat{\boldsymbol{x}})$, there is $g(\hat{\boldsymbol{x}}|\boldsymbol{x}) \geq g(\hat{\boldsymbol{x}}|\hat{\boldsymbol{x}})$ since $p_n(\hat{\boldsymbol{x}})$ gives the optimal y_n that minimizes the new objective. Interchanging the roles of $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ yields

$$g(\boldsymbol{x}|\hat{\boldsymbol{x}}) \geq g(\boldsymbol{x}|\boldsymbol{x}) = f(\boldsymbol{x}). \tag{11}$$

Combining (10) and (11) shows that $g(\boldsymbol{x}|\hat{\boldsymbol{x}})$ constitutes a surrogate function [27] with respect to $f(\boldsymbol{x})$. Thus, Algorithm 1 is in essence an MM procedure: In an alternating fashion, update $\boldsymbol{y}$ to construct a new upper bound on $f(\boldsymbol{x})$, then update $\boldsymbol{x}$ to improve the solution based on the current bound. By the MM theory [28], we further obtain the following result.

Proposition 2. *Algorithm 1 yields a monotonically decreasing convergence to a stationary point of* (2) *provided that the cost function is differentiable and strongly convex.*

4. FRACTIONAL PROGRAMMING FOR AOI

Consider a multiple-sources system wherein $K \geq 2$ source nodes share a common server as shown in Fig. 1. Each source node $k = 1, \ldots, K$ delivers update packets constantly toward the server at rate λ_k, while the departure rate of the queue awaiting the service is μ. The ith update packet from source k is delivered at time t_{ik} and finally departs the server at time t'_{ik}. The delay $t'_{ik} - t_{ik}$ is caused by the queue waiting and the server processing. At the current time τ, we use $\mathcal{N}_k(\tau)$ to denote the arrival time of the most recently received packet from source k:

$$\mathcal{N}_k(\tau) = \max\left\{t'_{ik} : t'_{ik} \leq \tau\right\}. \tag{12}$$

The instantaneous AoI of source node k at the present time τ, denoted by $\Delta_k(\tau)$, is defined to be the time elapsed since its

Table 1. Performance of the different rate control schemes

	Sum of AoI	Sum of Squared AoI
Max Rate	447.1	3.3×10^4
Equal Rate [29]	218.8	5.9×10^3
FP-Based Method	131.8	1.8×10^3

last update packet departs the server, i.e.,

$$\Delta_k(\tau) = \tau - \mathcal{N}_k(\tau). \tag{13}$$

As a result, $\Delta_k(\tau)$ increases linearly with τ, and drops whenever a new update packet departs the server, so $\Delta_k(\tau)$ has a sawtooth profile along the time axis as shown in Fig. 2. We are interested in the average AoI in the long run:

$$\bar{\Delta}_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Delta_k(\tau)d\tau, \tag{14}$$

which can be recognized as the average area of the trapezoid below each tooth of the sawtooth curve in Fig. 2.

The specific expression of $\bar{\Delta}_k$ depends on the queuing system model. For the M/M/1 model under LCFS-S, i.e., Last Come First Serve scheme that permits preempting package in Service, also with priority as considered in [29], the average AoI of source k is given by

$$\bar{\Delta}_k = \frac{1+\rho_k+3\hat{\rho}_k+3\hat{\rho}_k\rho_k+3\hat{\rho}_k^2+\hat{\rho}_k^2\rho_k+\hat{\rho}_k^3}{\mu\rho_k\left(1+\hat{\rho}_k\right)}, \tag{15}$$

where $\rho_k = \lambda_k/\mu$ and $\hat{\rho}_k = \sum_{i=1}^{k-1}\rho_i$. Let $U : \mathbb{R}^K \to \mathbb{R}$ be an increasing convex cost function of $\bar{\Delta}_k$. The rate control problem is to minimize an increasing convex cost function of AoI by optimizing λ_k's, i.e.,

$$\underset{\lambda_1,\ldots,\lambda_K}{\text{minimize}} \quad U(\bar{\Delta}_1,\ldots,\bar{\Delta}_K) \tag{16a}$$

$$\text{subject to} \quad 0 \le \lambda_k \le \mu, \text{ for } k = 1,\ldots,K, \tag{16b}$$

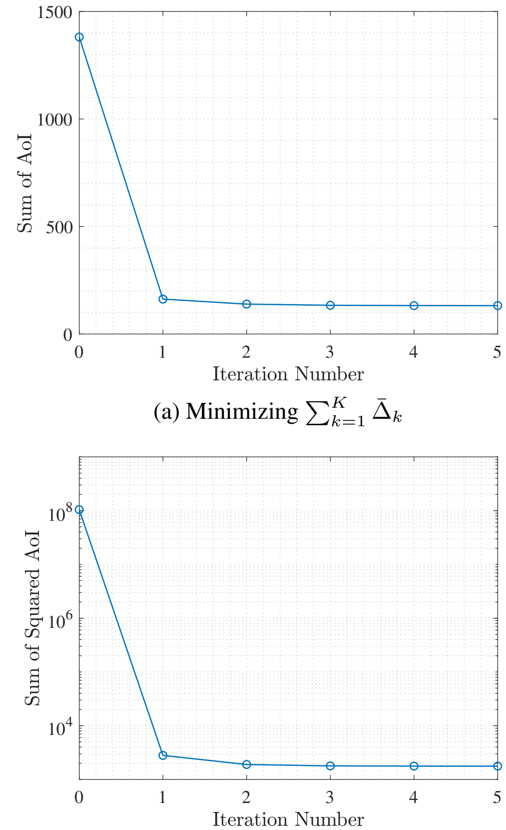
which encompasses the traditional sum-of-AoI minimization [22–26] as a special case. The following proposed method also works for other types of AoI penalization, e.g., the sum-of-squared-AoI minimization.

Although the fractional form of $\bar{\Delta}_k$ in (15) strongly suggests the use of FP, we cannot directly apply the technique in Corollary 1 because the numerator and denominator are not respectively convex and concave. Nevertheless, this issue can be addressed by rewriting $\bar{\Delta}_k$ as a sum of two fractions:

$$\bar{\Delta}_k = \frac{\hat{\rho}_k^2+3\hat{\rho}_k+1}{\mu(1+\hat{\rho}_k)} + \frac{(\hat{\rho}_k+1)^2}{\mu\rho_k}. \tag{17}$$

Now each ratio term has its numerator be convex and its denominator be concave. Notice that the objective remains an increasing convex function of these ratios.

We test the performance of the proposed inverse quadratic transform method by simulations. Let $K = 10$ and $\mu = 1$.



(a) Minimizing $\sum_{k=1}^{K}\bar{\Delta}_k$

(b) Minimizing $\sum_{k=1}^{K}\bar{\Delta}_k^2$

Fig. 3. Convergence of the proposed FP-based method.

Consider two benchmarks: (i) the equal rate optimization [29] that assumes all λ_k's are equal and then performs the one-dimensional search; (ii) the max rate policy that sets each $\lambda_k = \mu$. Two types of objectives are considered: the sum of AoI $\sum_k \bar{\Delta}_k$ and the sum of squared AoI $\sum_k \bar{\Delta}_k^2$. Observe from Fig. 3 that the proposed FP-based method has fast convergence for both sum-of-AoI minimization and sum-of-squared-AoI minimization. Actually, the majority of the AoI reduction is achieved after the first iteration. Moreover, as shown in Table 1, the proposed method outperforms the benchmark methods significantly. The sum-of-AoI of the proposed method is approximately 40% lower than that of the equal rate scheme [29], and is approximately 70% lower than that of the max rate scheme.

5. CONCLUSION

The paper proposes a new method called inverse quadratic transform for minimizing a sum of ratios, and more generally, for minimizing a cost function of multiple ratios. We also furnish a justification based on the MM theory. Furthermore, we illustrate the use of this new min-FP method in reducing AoI for multi-source networks.

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