Optimal Discrete Beamforming for Intelligent Reflecting Surface

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***Abstract*—This work pursues an efficient strategy of designing passive beamformer for intelligent reflecting surface (IRS) in order to maximize the overall channel strength. In particular, the choice of phase shift for each reflective element is restricted to $K \geq 2$ discrete values. Although the resulting discrete beamforming problem is believed to be NP-hard in some prior works, the paper shows that the global optimum of the binary case with $K = 2$ can actually be achieved in quadratic time. For a general K-ary beamforming problem with $K > 2$, the state-of-the-art polynomial time algorithm is to greedily project the relaxed solution (which is trivial) to the closest point in the constraint set. However, as shown in this paper, this greedy method is unbounded in the sense that its performance can be arbitrarily bad. In contrast, we propose a linear time algorithm that is capable of providing a near-optimal solution with an approximation ratio of $(1 + \cos(\pi/K))/2$, i.e., at least 75% of the global optimum can be reached when $K \geq 3$. Furthermore, inspired by the so-called RFocus method in [1], we develop a statistic implementation of the above approximation algorithm in the absence of channel state information (CSI).**

I. Introduction

Intelligent reflecting surface (IRS) is an emerging wireless device [2], [3] that uses a large array of passive "mirrors"—reflective elements to spatially concentrate the impinging radio waves on the intended receiver, thereby enhancing reliability and energy efficiency of wireless transmission. This work aims to maximize the overall channel strength toward the receiver by coordinating phase responses across the reflective elements, namely the IRS beamforming.

From a practical standpoint, we impose discrete limitations on the IRS beamformer such that the choice for the phase shift induced by each reflective element is restricted to $K \geq 2$ discrete values. We begin with the binary case wherein every phase shift is either 0 or π. This problem deceptively appears NP-hard because of its discrete constraint [2], [4], [5]. To clarify this misconception, we propose an exact algorithm that is capable of reaching the global optimum in quadratic time. For a general discrete beamforming problem with $K > 2$, we propose a linear time algorithm with a provable approximation ratio, whereas the greedy method [1], [6] can end up with arbitrarily bad performance. Furthermore, the proposed approximation algorithm can be carried out even when channel state information (CSI) is not available.

While many developments [7]–[9] have taken place in the area of channel estimation for IRS (although the recent work [10] argues that we could skip channel acquisition by means of machine learning), a large group of other works examine the beamforming aspect assuming that CSI has been obtained. By recognizing IRS as a multiple-passive-antenna equipment [1], a line of existing researches assume that the phase responses of reflective elements can be chosen arbitrarily, thus extending those conventional continuous beamforming algorithms for multiple-input multiple-output (MIMO) transmission to the IRS setting, e.g., singular value decomposition [11], semidefinite programming [12], [13], fractional programming [14], [15], and successive convex approximation [16]. Although the above works have shown that considerable gains can be achieved via continuous beamforming, the practical implementations [1], [2] would typically restrict the choices of phase shifts to a given set of discrete values, because of the hardware limitation as well as the economic concerns. Aimed at the global optimum of the discrete IRS beamforming, [17] and [5] use the exhaustive search and the branch-and-bound algorithm, respectively, both requiring exponential time.

In contrast, aimed at some reasonably good solution, the two recent works [1], [2] suggest projecting the continuous beamformer to the nearest discrete point in a greedy fashion. Specifically, with the continuous beamformer optimally aligning all the channels in one direction, this greedy method in [1], [2] for discrete beamforming would try to minimize the angle between the direct channel and each individual reflected channel. However, the paper demonstrates that the greedy approach in [1], [2] is unbounded in that we cannot find a positive constant to bound below the ratio between the solution of the greedy method and the global optimum.(Actually, the near-optimality claim in [1] is problematic.) In contrast, the proposed method guarantees a strictly positive approximation ratio of $(1 + \cos(\pi/K))/2$. But the idea of implementing the proposed approximation algorithm statistically without CSI stems from the so-called RFocus method in [1].

II. System Model

Consider a pair of transmitter and receiver, along with an IRS deployed to facilitate the wireless transmission between them. The IRS consists of N passive reflective elements, each introducing an independent reflected path from the transmitter toward the receiver. We use $h_0 \in \mathbb{C}$ to denote the channel of the direct path, and $h_n \in \mathbb{C}$ the channel of the nth reflected path for any $n = 1, \dots, N$. These channels can be alternatively expressed in an exponential form, i.e.,

$$h_n = \beta_n e^{j\alpha_n},\ n = 0, 1, \dots, N, \tag{1}$$

with the modulus $\beta_n > 0$ and the phase $\alpha_n \in [0, 2\pi)$. These channels are all fixed throughout our discussion.

Let $\theta_n \in [0, 2\pi]$ be the phase shift induced by the nth reflective element in its incident radio wave, so the overall channel strength from the transmitter to the receiver can be computed as a function of the beamforming vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$:

$$f(\boldsymbol{\theta}) = \left| h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n} \right|^2. \tag{2}$$

Furthermore, we assume that every phase shift θ_n must be selected from a uniform discrete set

$$\Phi_K = \big\{0, \omega, \cdots, (K-1)\omega\big\}, \tag{3}$$

where

$$\omega = \frac{2\pi}{K} \tag{4}$$

for a positive integer $K \geq 2$. We seek the optimal beamformer $\boldsymbol{\theta}$ to maximize the overall channel magnitude, i.e.,

$$\underset{\boldsymbol{\theta}}{\text{maximize}} \quad f(\boldsymbol{\theta}) \tag{5a}$$

$$\text{subject to} \quad \theta_n \in \Phi_K,\ n = 1, \dots, N. \tag{5b}$$

The above problem is difficult to solve in general because of the discrete constraint. Besides, the lack of the CSI $\{h_0, h_1, \dots, h_N\}$ in practice would cause another challenge of the discrete beamforming design for IRS. In the rest of the paper, we first focus on solving for $\boldsymbol{\theta}$ with perfect CSI, then discuss how to deal with the absence of CSI in Section V.

III. Binary Beamforming When $K = 2$

As the number of phase-shift candidates K tends to infinity, the discrete beamforming problem in (5) reduces to a continuous problem for which the solution is trivial:

$$\tilde{\theta}_n = \alpha_0 - \alpha_n,\ n = 1, \dots, N. \tag{6}$$

It is tempting to think that the difficulty of the problem decreases with K. In particular, one may believe that the binary case with $K = 2$ is the hardest and even NP-hard.

However, in this section we present a counter-intuitive result that the binary beamforming problem is polynomial time solvable, as stated in the following proposition.

Proposition 1 (Globally Optimal binary Beamforming): The binary beamforming problem in (5) with $K = 2$ can be optimally solved in $O(N^2)$ time.

Proof: Note that each $\theta_n \in \{0, \pi\}$ when $K = 2$. Introduce a binary vector $\boldsymbol{x} = (1, x_1, \dots, x_N)^T \in \{-1, 1\}^{N+1}$, where $x_n = \cos(\theta_n)$ for $n = 1, \dots, N$. Using $\boldsymbol{x}$ to substitute $\boldsymbol{\theta}$ in $f(\boldsymbol{\theta})$ cast the objective function to

$$f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x}, \tag{7}$$

where $\boldsymbol{C} = \big[c_{ij}\big]$ is an $(N{+}1)\times(N{+}1)$ real symmetric matrix with each entry $c_{ij} = \Re\big\{h_i \overline{h}_j\big\}$, $i = 0, \dots, N$, $j = 0, \dots, N$. Observe that $\mathrm{rank}(\boldsymbol{C}) \leq 2$. We shall consider the optimal $\boldsymbol{x}^\star$ for the rank one case and the rank two case respectively, then recover the optimal phase shift as $\theta_n^\star = \arccos(x_n^\star)$.

If $\boldsymbol{C}$ is rank one, then it can be decomposed as $\boldsymbol{C} = \lambda \boldsymbol{v}\boldsymbol{v}^T$ with a unique positive eigenvalue $\lambda > 0$ and the corresponding eigenvector $\boldsymbol{v} = (v_0, v_1, \dots, v_N)^T \in \mathbb{R}^{N+1}$. To maximize $f(\boldsymbol{x}) = \lambda \boldsymbol{x}^T \boldsymbol{v}\boldsymbol{v}^T \boldsymbol{x}$, we simply choose $x_n \in \{-1, 1\}$ to make $x_n v_n$ have the same sign as $x_0 v_0$, i.e., $x_n^\star = \mathrm{sgn}(v_0 v_n)$.

If $\boldsymbol{C}$ is rank two, we have the decomposition $\boldsymbol{C} = \boldsymbol{V}^T \boldsymbol{V}$ where $\boldsymbol{V} \in \mathbb{R}^{2\times(N+1)}$ is row full rank. With the jth column of $\boldsymbol{V}$ denoted by $\boldsymbol{v}_j \in \mathbb{R}^2$, $j = 1, \dots, N+1$, we narrow down the search space of $\boldsymbol{x}$ to

$$\mathcal{X} = \Big\{ \boldsymbol{x} \,\Big|\, \exists\, \boldsymbol{z}_j \in \mathbb{R}^2 \text{ s.t. } \boldsymbol{z}_j^T \boldsymbol{v}_j = \mathrm{sgn}(x_{j-1}), \quad \forall j = 1, \dots, N+1 \Big\}. \tag{8}$$

According to a *zonotype* argument in [18], the above set $\mathcal{X}$ must contain the optimal $\boldsymbol{x}^\star$. Most importantly, $|\mathcal{X}| = 2N$ so $\boldsymbol{x}^\star$ can be found in $O(N)$ time.

Moreover, it takes $O(N^2)$ time to compute $\boldsymbol{C}$ and $\mathcal{X}$, so the binary beamforming problem is globally solvable in $O(N^2)$ for either rank one or rank two case. ■

In contrast, the greedy method in [1], [6] simply rounds the relaxed solution $\tilde{\theta}$ in (6) to the closest point in Φ_K, i.e.,

$$\theta_n' = \arg \min_{\theta_n \in \Phi_K} \big|\theta_n - \tilde{\theta}_n\big|. \tag{9}$$

We use the following example to show that the greedy method in [1], [6] can lead to arbitrarily bad performance.

Example 1: For the reflected channels, assume that $\beta_1 = \beta_2 = \cdots = \beta_N$ and assume that half of them have the phase $\alpha_n = \alpha_0 - \delta + \pi/2$ while the other half have $\alpha_n = \alpha_0 + \delta - \pi/2$ given some $0 < \delta < \pi/2$, so the greedy method sets every θ_n' to 0. As a result, $f(\boldsymbol{\theta}') \to 0$ as $\beta_0 \to 0$ and $\delta \to 0$.

IV. General K-Ary Beamforming

The main results of this section are twofold: an exact algorithm that is far more efficient than exhaustive search, and a linear time algorithm with guaranteed performance.

A. Exact Algorithm via Sectorization

We propose a sectorizing scheme that can reduce the size of the search space on $\boldsymbol{\theta}$ from K^N to 2^N. Consider the following four sectors around h_0 on the complex plane:

$$\mathcal{S}_i = \left\{ x \in \mathbb{C} \,\middle|\, \alpha_0 + \frac{(2-i)\omega}{2} \leq \arg(x) < \alpha_0 + \frac{(3-i)\omega}{2} \right\}, \quad i = 1, 2, 3, 4. \tag{10}$$

Fig. 1 illustrates the above sectorization. We use $h^\star$ to denote the overall channel corresponding to an optimal $\boldsymbol{\theta}^\star$, i.e.,

$$h_\Sigma^\star = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n^\star}. \tag{11}$$

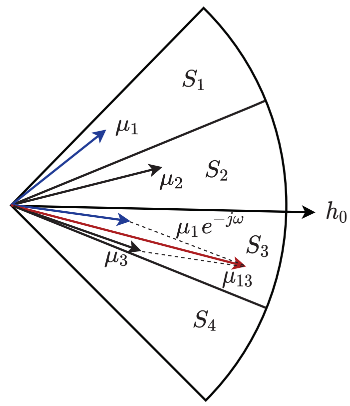

Fig. 1. Four sectors $\mathcal{S}_1$ to $\mathcal{S}_4$. In the proof of Proposition 3, we rotate μ_1 clockwise by an angle of ω, then further combine it with μ_3 to obtain μ_{13}.

Although the optimal solution of $\boldsymbol{\theta}$ may not be unique, at least one $\boldsymbol{\theta}^\star$ has its $h^\star_\Sigma$ close to h_0, as stated in the following lemma.

Lemma 1: There exists at least one optimal $\boldsymbol{\theta}^\star$ that renders overall channel $h^\star_\Sigma$ fall in either Sector $\mathcal{S}_2$ or Sector $\mathcal{S}_3$.

With $h_n e^{j\theta^\star_n}$ restricted to $\mathcal{S}_2$ and $\mathcal{S}_3$, we can shrink the range of possible solutions for $\boldsymbol{\theta}$ as in the following proposition.

Proposition 2: Given a subset $\mathcal{A} \subseteq \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, the function $\Theta(\mathcal{A})$ outputs a subset of Φ_K^N as

$$\Theta(\mathcal{A}) = \left\{\boldsymbol{\theta} \in \Phi_K^N \;\middle|\; h_n e^{j\theta_n} \in \mathcal{A},\ \forall n = 1, \ldots, N\right\}. \quad (12)$$

For the problem in (5), at least one optimal solution $\boldsymbol{\theta}^\star$ is contained in either $\Theta(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3)$ or $\Theta(\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4)$.

Proof: If $h^\star_\Sigma \in \mathcal{S}_2$, then the optimal $\theta^\star_n$ would rotate h_n to the closest possible position to $\mathcal{S}_2$, so $h_n e^{j\theta^\star_n} \in \Theta(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3)$. Likewise, $h_n e^{j\theta^\star_n}$ must be contained in $\Theta(\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4)$ if $h^\star_\Sigma \in \mathcal{S}_3$. The proof is then completed because $h^\star_\Sigma$ must reside in either $\mathcal{S}_2$ or $\mathcal{S}_3$ according to Lemma 1. ■

In light of the above proposition, we decide the solution as

$$\boldsymbol{\theta}^\star = \arg\max_{\boldsymbol{\theta} \in \Omega_3} f(\boldsymbol{\theta}), \quad (13)$$

where the search space Ω_3 is given by

$$\Omega_3 = \Theta(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3) \cup \Theta(\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4). \quad (14)$$

Observe that $|\Omega_3| = 2^N$ which is much smaller than original search space size of K^N.

B. Efficient Approximation Algorithm

The central premise behind the above exact algorithm is that the set of reflected channels $\{h_n e^{j\theta^\star_n}\}$ under optimal beamforming can span at most three consecutive sectors. This causes the search space Ω_3 to be exponentially large. We wish to further narrow down the search space of $\boldsymbol{\theta}^\star$. A simple idea is to confine the span of $\{h_n e^{j\theta^\star_n}\}$ to just two consecutive sectors, so the search space Ω_3 becomes

$$\Omega_2 = \Theta(\mathcal{S}_1 \cup \mathcal{S}_2) \cup \Theta(\mathcal{S}_2 \cup \mathcal{S}_3) \cup \Theta(\mathcal{S}_3 \cup \mathcal{S}_4). \quad (15)$$

We then propose searching through Ω_2, i.e.,

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta} \in \Omega_2} f(\boldsymbol{\theta}). \quad (16)$$

The above search is superfast since $|\Omega_2| \le 3$. Besides, it takes $O(N)$ time to compute the search space Ω_2, the overall algorithm runs in $O(N)$ time.

We may miss the optimal solution $\boldsymbol{\theta}^\star$ when switching from Ω_3 to Ω_2, but the following proposition shows that the suboptimal solution by Ω_2 is close to the global optimum.

Proposition 3 (Near-Optimal K-Ary Beamforming): A general discrete beamforming problem in (5) with $K \ge 2$ can be approximately solved in $O(N)$ time, such that the solution $\hat{\boldsymbol{\theta}}$ and the global optimum $\boldsymbol{\theta}^\star$ satisfy

$$\frac{1 + \cos(\pi/K)}{2} f(\boldsymbol{\theta}^\star) \le f(\hat{\boldsymbol{\theta}}) \le f(\boldsymbol{\theta}^\star). \quad (17)$$

Proof: Without loss of generality, assume that an optimal overall channel $h^\star_\Sigma$ lies in $\mathcal{S}_2$. We partition $h^\star_\Sigma$ into three terms:

$$h^\star_\Sigma = \mu_1 + \mu_2 + \mu_3, \quad (18)$$

where μ_i refers to the sum of channels (either direct or reflected) in sector $\mathcal{S}_i$. Let

$$\mu_{13} = \mu_1 e^{-j\omega} + \mu_3 \quad (19)$$

as illustrated in Fig. 1. We then bound $f(\boldsymbol{\theta}^\star)$ as

$$\begin{aligned} f(\boldsymbol{\theta}^\star) &= \big|\mu_1 + \mu_2 + \mu_3\big|^2 \\ &\le \big(\big|\mu_1 + \mu_3\big| + \big|\mu_2\big|\big)^2 \\ &\le \big(\big|\mu_{13}\big| + \big|\mu_2\big|\big)^2. \end{aligned} \quad (20)$$

In the meanwhile, since $h^\star_\Sigma$ is assumed to be in $\mathcal{S}_2$, the solution $\hat{\boldsymbol{\theta}} \in \Omega_2$ is contained in either $\Theta(\mathcal{S}_1 \cup \mathcal{S}_2)$ or $\Theta(\mathcal{S}_2 \cup \mathcal{S}_3)$, and hence $f(\hat{\boldsymbol{\theta}})$ can be bounded as

$$\begin{aligned} f(\hat{\boldsymbol{\theta}}) &= \max\left\{\big|\mu_1 e^{-j\omega} + \mu_2 + \mu_3\big|^2, \big|\mu_1 + \mu_2 + \mu_3 e^{j\omega}\big|^2\right\} \\ &= \max\left\{\big|\mu_{13} + \mu_2\big|^2, \big|\mu_{13} e^{j\omega} + \mu_2\big|^2\right\} \\ &\ge |\mu_2|^2 + |\mu_{13}|^2 + 2\big|\mu_2 \mu_{13}\big| \cos(\omega/2), \end{aligned} \quad (21)$$

where the last step follows since $\max\{|\mu_{13} + \mu_2|^2, |\mu_{13} e^{j\omega} + \mu_2|^2\}$ is minimized when μ_2 is a bisector of the angle between μ_{13} and $\mu_{13} e^{j\omega}$.

We introduce an auxiliary variable

$$\lambda = \frac{|\mu_{13}|}{|\mu_2|}. \quad (22)$$

Taken together, the bounds in (20) and (21) yield

$$\frac{f(\boldsymbol{\theta}^\star)}{f(\hat{\boldsymbol{\theta}})} \le \frac{(\lambda + 1)^2}{\lambda^2 + 1 + 2\lambda \cos(\omega/2)} \quad (23a)$$

$$\le \frac{2}{1 + \cos(\omega/2)} \quad (23b)$$

with the second equality if and only if $\lambda = 1$. Finally, plugging $\omega = 2\pi/K$ into (23b) verifies the approximation ratio. ■

Corollary 1: As $K \to \infty$, the approximation algorithm in (16) reduces to the relaxed solution in (6) and $f(\hat{\boldsymbol{\theta}}) = f(\boldsymbol{\theta}^\star)$.

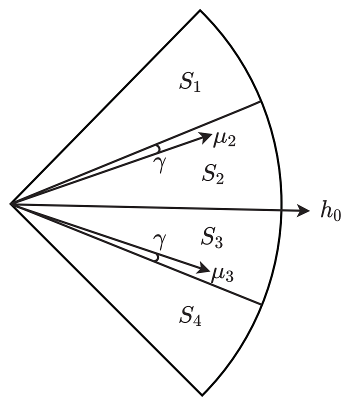

Fig. 2. The worst-case scenario of the greedy method. If $\mu_1 = 0$ and $|\mu_2| = |\mu_3|$, the overall channel stength tends to $4|\mu_2|^2 \cos(\pi/K)$ as the angle $\gamma \to 0$ and $\beta_0 \to 0$, while the global optimum tends to $4|\mu_2|^2$.

C. Other Approximation Algorithms

The aforementioned greedy method in (9) can be directly applied for the general K-ary beamforming as in [1], [6].

Proposition 4: The greedy method in (9) produces a solution $\boldsymbol{\theta}'$ that satisfies

$$\cos^2(\pi/K) f(\boldsymbol{\theta}^\star) \le f(\boldsymbol{\theta}') \le f(\boldsymbol{\theta}^\star). \tag{24}$$

Proof: Clearly, $f(\boldsymbol{\theta}^\star) \le \left(\sum_{n=0}^N \beta_n\right)^2$. By projecting each $h_n e^{j\theta_n}$ onto h_0, we obtain a lower bound as $f(\boldsymbol{\theta}') \ge \left(h_0 + \sum_{n=1}^N \beta_n \cos(\theta'_n - \tilde{\theta}_n)\right)^2 \ge \left(\sum_{n=0}^N \beta_n \cos(\pi/K)\right)^2$, thus establishing the approximation ratio of $\cos^2(\pi/K)$. ■

It is worth pointing out that the lower bound in (24) is sharp as illustated in Fig. 2.

Another approach is to rewrite the objective function $f(\boldsymbol{\theta})$ as $f(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{C} \boldsymbol{x}$ with $\boldsymbol{C} \succcurlyeq \boldsymbol{0}$; this reformulation is akin to the binary case in (7) except that $\boldsymbol{x}$ and $\boldsymbol{C}$ are now both complex-valued. This complex K-ary quadratic program (QP) is approximately solvable by the semidefinite relaxation (SDR) technique [19], [20], as stated in the following proposition.

Proposition 5 (SDR Method [19], [20]): The discrete beamforming problem in (5) can be recognized as a complex K-ary QP, and can be further recast to a convex optimization problem by the SDR method. The resulting solution $\boldsymbol{\theta}''$ satisfies

$$\frac{(k\sin(\pi/K))^2)}{4\pi} f(\boldsymbol{\theta}^\star) \le f(\boldsymbol{\theta}'') \le f(\boldsymbol{\theta}^\star). \tag{25}$$

We compare the approximation ratios of the various algorithms in Fig. 3. It shows that the proposed algorithm provides better approximation accuracy than the greedy method and the SDR method given any value of $K \ge 2$.

V. Beamforming Without CSI

The discussion in the paper thus far assumes perfect CSI. We now provide a blind beamforming scheme that is capable of approaching the approximation algorithm in the previous section asymptotically when CSI is not available.

We start by considering how to implement the greedy method in (9) in the absence of CSI. The main idea follows [1] closely. Let us fix the transmit signal at 1 and randomly choose each phase shift θ_n from Φ_K in an i.i.d. fashion. With the thermal noise $z \sim \mathcal{CN}(0, \sigma_0^2)$, the received signal y is

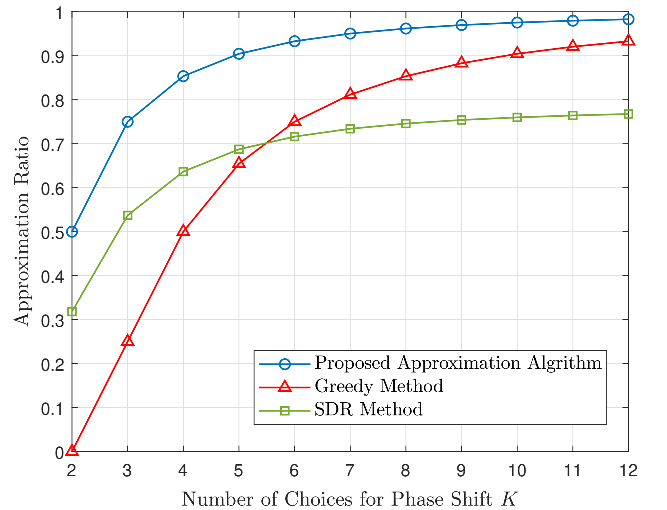

Fig. 3. The approximation ratios of various algorithms.

$$y = \left(h_0 + \sum_{n=1}^N h_n e^{j\theta_n}\right) \cdot 1 + z. \tag{26}$$

Every tuple of $(\boldsymbol{\theta}, z, y)$ is referred to as a random sample in which only $\boldsymbol{\theta}$ and y are known. For each $\vartheta \in \Phi_K$ and each $n = 1, \ldots, N$, evaluate the sample mean of the received signal power $|y|^2$ conditioned on $\theta_n = \vartheta$, denoted by

$$Q_n(\vartheta) = \mathbb{E}_{\boldsymbol{\theta},z}\left[|y|^2 \,|\, \theta_n = \vartheta\right],\ n = 1, \ldots, N. \tag{27}$$

It is a natural idea to decide the phase shift θ_n according to the "average" performance $Q_n(\vartheta)$; this choice of θ_n is denoted by

$$\psi_{23,n} = \arg\max_{\vartheta \in \Phi_K} Q_n(\vartheta),\ n = 1, \ldots, N. \tag{28}$$

It turns out that the above scheme is asymptotically equivalent to the greedy method as shown in the following proposition.

Proposition 6 (Greedy Method Without CSI): As the number of random samples $T \to \infty$ and the number of reflective elements $N \to \infty$, the beamforming scheme in (28) approaches the greedy method in (9).

Proof: We focus on showing that $\psi_{23,n}$ in (28) tends to θ'_n in (9). If θ_n has been fixed at $\psi_{23,n}$ while the rest phase shifts θ_m are chosen randomly, then the reflected component $\sum_{m \ne n} h_n e^{j\theta_n}$ amounts to a Gaussian random variable of $\mathcal{CN}(0, \sigma_1^2)$ as $T \to \infty$ and $N \to \infty$, which is independent of another Gaussian random variable $z \sim \mathcal{CN}(0, \sigma_0^2)$. With the fixed component $h_0 + h_n e^{j\theta_n}$ and the Gaussian random component $\sum_{m \ne n} h_n e^{j\theta_n} + z$, the received signal envelop $|y|$ has a Rician distribution, so the average power $|y|^2$ can be computed as $|h_0 + h_n e^{j\theta_n}|^2 + \sigma_0^2 + \sigma_1^2$. Clearly, the average power is maximized when $h_n e^{j\theta_n}$ is at the closest position to h_0, namely the greedy method. ■

Recall that the greedy method basically aims to rotate every reflected channel to the inside of $\mathcal{S}_2 \cup \mathcal{S}_3$. In contrast, the proposed approximation algorithm additionally entails rotating every reflected channel into $\mathcal{S}_1 \cup \mathcal{S}_2$ and into $\mathcal{S}_3 \cup \mathcal{S}_4$. These two types of rotation can be readily performed if we can tell which of $\{h_n e^{j\psi_{23}}\}$ lie in $\mathcal{S}_3$ under the greedy method.

Toward this end, we first compute

$$g_0 = \mathbb{E}_{\boldsymbol{\theta},z}\big[y\big] \tag{29}$$

and

$$g_n = \mathbb{E}_{\boldsymbol{\theta},z}\big[y \,|\, \theta_n = \psi_{23,n}\big],\; n = 1,\ldots,N, \tag{30}$$

then decide which of $\mathcal{S}_2$ and $\mathcal{S}_3$ each $h_n e^{j\psi_{23,n}}$ belongs to as

$$\mu_n = \begin{cases} 0 & \text{if } \mathrm{Im}\{g_0 - g_n\} < 0; \\ 1 & \text{otherwise;} \end{cases} \quad n = 1,\ldots,N, \tag{31}$$

where the indicator variable μ_n equals to 0 if $h_n e^{j\psi_{23,n}} \in \mathcal{S}_2$ and equals to 1 if $h_n e^{j\psi_{23,n}} \in \mathcal{S}_3$.

Thus, in order to render all the reflected channels in $\mathcal{S}_1 \cup \mathcal{S}_2$, we simply rotate those $h_n e^{j\psi_{23,n}}$ with $\mu_n = 1$ counterclockwise by an angle of ω, so the corresponding phase shift is

$$\psi_{12,n} = \psi_{23,n} + \mu_n \omega,\; n = 1,\ldots,N. \tag{32}$$

Furthermore, the phase shifts that rotate the reflected channels into $\mathcal{S}_1 \cup \mathcal{S}_2$ are given by

$$\psi_{34,n} = \psi_{12,n} - \omega,\; n = 1,\ldots,N. \tag{33}$$

Let $\boldsymbol{\psi}_{12} = (\psi_{12,1},\ldots,\psi_{12,N})$, and define $\boldsymbol{\psi}_{23}$ and $\boldsymbol{\psi}_{34}$ similarly. We propose to decide the beamformer as

$$\hat{\hat{\boldsymbol{\theta}}} = \arg \max_{\boldsymbol{\vartheta} \in \{\boldsymbol{\psi}_{12},\boldsymbol{\psi}_{23},\boldsymbol{\psi}_{34}\}} \mathbb{E}_z\big[|y|^2 | \boldsymbol{\theta} = \boldsymbol{\vartheta}\big]. \tag{34}$$

Clearly, the above beamforming method without CSI can approach the approximation algorithm in Section IV asymptotically, as summarized in the following proposition.

Proposition 7 (Approximation Algorithm Without CSI): As the number of random samples $T \to \infty$ and the number of reflective elements $N \to \infty$, beamforming method in (34) approaches the proposed approximation algorithm in (16).

VI. Simulation Results

We now validate the performance of the proposed algorithms in simulations. Following [6], we fix the transmit power at 20 dBm, and assume that each pathloss component $\beta_n = 10^{-3} \cdot d^{-3.5}$ where the distance $d = 100$ meters in our case. The background noise power σ^2 is set to 10^{-9} mW. Moreover, we assume Rayleigh fading by drawing the channel phase variables $(\alpha_0, \alpha_1, \ldots, \alpha_N)$ randomly from the uniform distribution $U[0, 2\pi)$ in an i.i.d. fashion.

Fig. 4 compares the average received signal power across the worst 1% random trials of the binary beamforming case when CSI is known precisely. We first remark that the global optimum can always be obtained efficiently by the proposed $O(N)$-time exact algorithm. Observe that the proposed approximation algorithm is quite close to the global optimum achieved by the proposed exact algorithm given any value of N. The actual gap between them is even smaller than the theoretical guarantee as suggested by the approximation ratio. Observe also that the gap between the approximation algorithm and the greedy method [1], [6] increases with the number of reflective elements. In particular, when $N = 300$, the approximation algorithm improves upon the greedy method by more than 16%. Thus, we conclude that the proposed approximation algorithm is more suited for an IRS with massive reflective elements. We further include a baseline algorithm called random method—which tries out 100 random choices of $\boldsymbol{\theta}$ and uses the best one. The figure shows that taking 100 random samples for the random method is insufficient even if we have only 50 reflective elements.

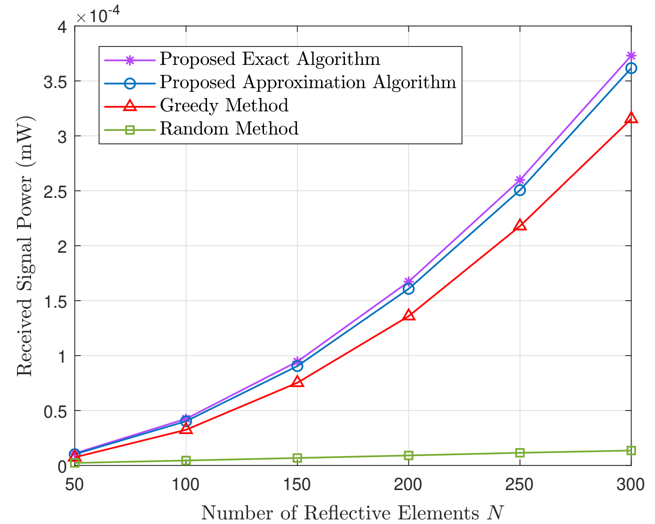

Fig. 4. The received signal power vs. the number of reflective elements when $K = 2$ and CSI is known.

We now look at the cases where CSI is not available. With respect to the binary beamforming problem, Fig. 5 compares the cumulative distribution functions (CDFs) of the received signal power for the greedy method without CSI and the proposed approximation algorithm without CSI in Section V. It can be seen that the approximation algorithm outperforms the greedy method especially in the low-percentile regime, i.e., the approximation algorithm provides more robust optimization against the worst-case scenario. Observe that both of these algorithms have improved performance when the number of random samples T is increased.

Furthermore, we plot the CDF curves of the two algorithms without CSI in Fig. 6 with $K = 4$. It is worthwhile to remark that the two algorithms with $T = 100$ actually become even worse when K is raised from 2 to 4, as can be seen by comparing Fig. 5 and Fig. 6. In principle, the larger K is, the more sensitive the two algorithms are to the channel estimation error. Thus, the insight we gain here is that increasing K does not necessarily enhance the performance when the number of random samples is limited. In contrast, when T is increased to 1000, the algorithms with $K = 4$ start to outperform the binary beamforming significantly. The figure also shows that

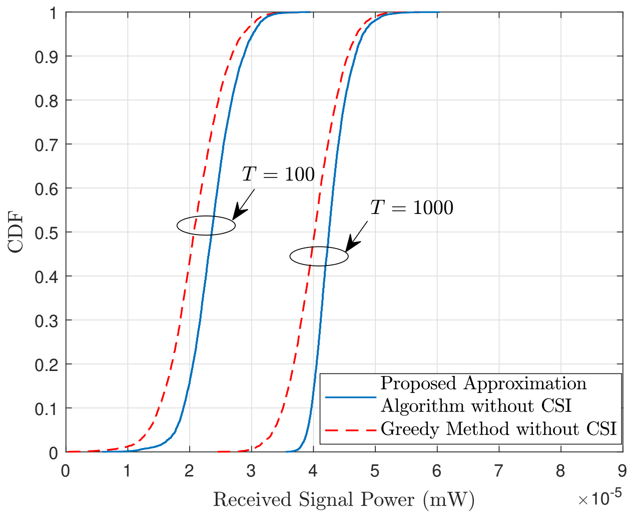

Fig. 5. CDF of received signal power with $K = 2$ when CSI is unknown.

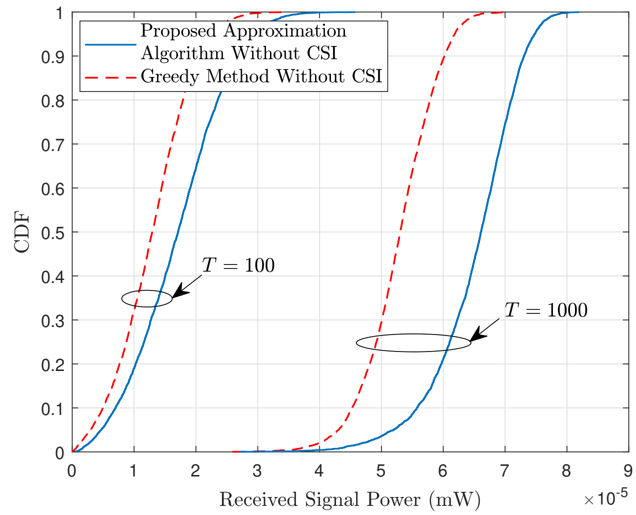

Fig. 6. CDF of received signal power with $K = 4$ when CSI is unknown.

the proposed approximation algorithm is much better than the greedy method when $T = 1000$, e.g., the received signal power of the approximation algorithm is over 25% higher than that of the greedy method at the 30th percentile.

VII. Conclusion

Despite the forbidding discrete constraints in the IRS beamforming problem, this work shows that the global optimum can obtained in quadratic time when the constraint is binary. For a general discrete beamforming problem, we propose a linear time algorithm that has near-optimal performance with guaranteed approximation accuracy, as compared to the existing greedy method [1], [6] that can lead to arbitrarily bad solution. We further propose a statistic implementation of this algorithm without CSI. Simulation results show that the proposed algorithms yield far more robust optimization than the baseline methods.

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